UIUCDCS-R-78-947 UILU-ENG 78 1739

DEFICIENT GENERALIZED FIBONACCI MAXIMUM PATH GRAPHS

BY

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November 1978

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October 1978

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The work of this author was supported in part by the National Science Foundation under grant NSF MCS 77-22830.

TABLE OF CONTENTS

ABSTRACT

The structure of an acyclic directed graph with n vertices and m edges, maximizing the number of distinct paths between two given vertices, is studied. New techniques for solving this problem are developed, thus enabling us to extend previous results.

I. INTRODUCTION

1.1 In this work we continue to investigate the structure of an acyclic directed graph with n vertices and m edges, maximizing the number of distinct paths between two given vertices.

In previous work it was shown that there exists such a graph containing a Hamiltonian path, thus uniquely ordering the vertices. The length of an edge (i,j) is defined now as $k=j-i$ and it belongs to level k. It was further shown that such a graph contains k-1 full levels and some edges of level k - a deficient k-generalized Fibonacci graph.

1.2 We investigate the distribution of the edges in level k. In particular, we find that even for $k=3$ the structure is quite complicated. The behaviour of the solution is different in three ranges of the number of the edges (called "phases"). We show some forbidden configurations for the solution. These observations, combined with some local properties, lead to the solution for the different phases. The proof techniques which were developed can help to investigate the more complicated distribution of the edges in levels k, $k>3$.

1.3 Section 2 presents the problem, and the solution is summarized in section 3. The analysis of the solution relies on certain local properties studied in section 4. The solution is partitioned into three phases, and these phases 1, 2 and 3 are the subject of sections 5, 6 and 7, respectively. General description of the proof techniques appear in the beginning of section 5.

II. PRESENTATION OF THE PROBLEM

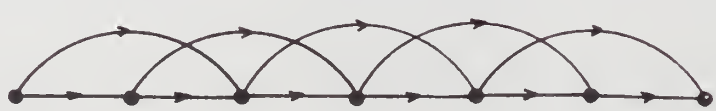
2.1 Let G be an acyclic directed graph without multiple edges or isolated vertices, containing two distinguished vertices s and t. Define N(G) to be the number of distinct paths from s to t. We consider the following problem: given integers n and m, find a graph G with n vertices and m edges maximizing N(G). We call such a graph G a maximum path graph and define $N_{n,m}=N(G)$. Clearly $N_{n,m}$ is defined in the domain $n-1 \leq m \leq \binom{n}{2}$.

2.2 In Perl [1] several cases of maximum path graphs are studied, given only the number of edges m. It is shown that for acyclic directed graphs without multiple edges the (almost) Fibonacci graphs are maximum path graphs for (even) odd number of edges. Examples are shown in figure 1. It is clear that for a (almost) Fibonacci graph G of n vertices $N(G)=F_n$ ($N(G)=2F_{n-2}$), where $\{F_n\}$ is the Fibonacci sequence defined by

$$F_1 = F_2 = 1 , \qquad F_{i+1} = F_i + F_{i-1} \qquad \text{for } i > 1.$$

Hence $N_{n,2n-3} = F_n$ and $N_{n,2n-4} = 2F_{n-2}$.

The Fibonacci graph for n=7



The almost Fibonacci graph for n=7

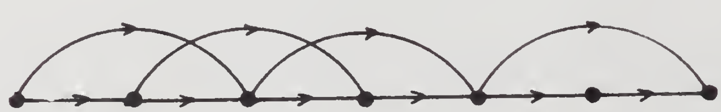


Figure 1

2.3 In Golumbic and Perl [2] maximum path acyclic directed graphs are studied, given the number of vertices n and the number of edges m. It is shown that there exists such a maximum path graph containing a Hamiltonian path, thus uniquely ordering the vertices $\{1,2,...n\}$. Hence, all the edges are of the form (i,j), $i<j$. The length of an edge (i,j) is $k=j-i$, and it belongs to level k. The number of distinct paths from vertex 1 to vertex i is denoted by $P(i)$, and $N(G)=P(n)$. It is further shown that if a maximum path graph contains an edge of level k then all the previous levels are full; i.e. all the edges of these levels belong to the graph.

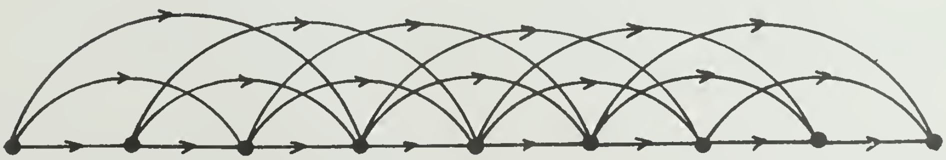
Therefore, either the graph contains exactly k full levels, or k-1 full levels and some edges of level k. A graph of the first kind is called a k-generalized Fibonacci graph, since the nimber of paths it contains is F_n^k, the n-th k-generalized Fibonacci number, defined by

$$F_i^k = \begin{cases} 1 & i=1,2\ , \\ \sum_{j=1}^{i-1} F_j^k & i<k\ , \\ \sum_{j=i-k}^{i-1} F_j^k & i \geq k\ . \end{cases}$$

A graph of the second kind is called deficient k-generalized Fibonacci graph. Examples are shown in figure 2.

A 3-generalized Fibonacci graph

(1)



A deficient 3-generalized Fibonacci graph

(2)

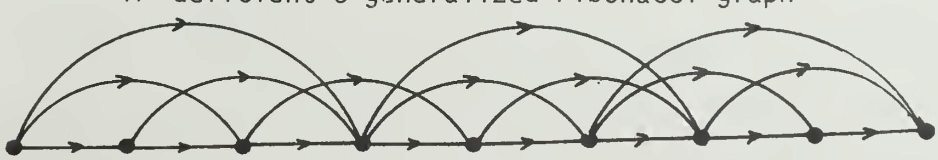


Figure 2

2.4 In this paper we investigate the distribution of the edges of level k in a deficient k-generalized Fibonnaci graph. We refer to such a graph as the solution. The solution for k=2 is given in [2] . We find that even for k=3 the structure is quite complicated, and this work is dealing only with this case. The solution for k>3 seems to be more complicated, but of the same spirit as the one for k=3. Thus the understanding and the proof techniques which were developed here can help to investigate the solution for k>3.

For convenience we will refer and draw only edges of level 3, and also describe level 3 by a 0,1-sequence $c_1c_2...c_{n-3}$, where c_i=1 if (i,i+3) is an edge and 0 otherwise. For example, the sequence 100101 describes the graph of figure 2(2). A compact representation of this binary sequence is used whenever possible. For example, $(100)^3 01(011)^2$ means 1001001000101011011. [x] means that x is optional; for example, $10[0](100)^k$ represents both $100(110)^k$ and $10(110)^k$.

III. DESCRIPTION OF THE SOLUTION

3.1 Let us note first that the solution is not unique, and thus we only claim that the graph described is a solution. The behaviour of the solution is different for three ranges, where the number of edges in level 3 is about (1) up to one third, (2) between one third and two thirds, and (3) more than two thirds of the n-3 possible edges in this level. These ranges are called phase 1, phase 2 and phase 3, respectively.

3.2 First we mention local properties of the solution. Let I denote the number of edges in level 3, $0 < I \leq n-3$. If $I>1$ then the solution contains both end-edges (1,4) and (n-3,n). Furthermore, if $I \leq n-5$ then there exists a solution not containing the near-end edges (2,5) and (n-4,n-1). For example, the solution for n=9, I=3 (figure 2(2))satisfies both properties.

Two edges are disjoint if they have no common interval. A run of consecutive disjoint edges is described by the configuration $(001)^k00$, $k>1$. If a graph does not contain the edge (i,j) then (i,j) is a non-edge. If the graph contains neither of (i,i+3), (i+1,i+4) and (i+2,i+5) then (i+2,i+3) is a gap.

3.3 We describe the three phases of the solution.

Phase 1: $I \leq \left\lfloor \frac{n}{3} \right\rfloor$.

($\lfloor t \rfloor$ denotes the largest integer not larger than t.)

The graph is composed of two runs of consecutive disjoint edges, one starting and the other ending with end-edges. The number of edges in each run is arbitrary. In case $I=(n-1)/3$, where $n \equiv 1 \pmod 3$, the two runs merge into one.

Examples: $(100)^4 0^2 (001)^2$, $(100)^4 1$.

Phase 2: $\left\lfloor \frac{n+2}{3} \right\rfloor \leq I \leq \left\lfloor \frac{2n-6}{3} \right\rfloor$.

In case $n \not\equiv 1 \pmod 3$ the solution is composed of two runs of consecutive disjoint edges, one starting and the other ending with both end-edges. The number of the edges in the two runs is arbitrary, except that

the two near-end edges are not contained in a run.

Examples: $100(110)^4 01$ (which is composed of $(100)^5 0^2$ and $0^2(001)^5$), $(100)^2 10(110)^2 1$.

In case $n \equiv 1 (\bmod 3)$ the solution is composed of two runs of consecutive disjoint edges, one starting and ending with both end-edges, and the other run starting at any vertex but not containing a near-end edge.

Examples: $(100)^2(110)^2 1$, $10010(110)^3 01$.

There are two border cases discussed later.

Phase 3: $I \geq 2 \left\lfloor \frac{n}{3} \right\rfloor - 1$.

The structure of the graph is quite complicated in terms of runs of consecutive disjoint edges. On the other hand, the solution contains all n-3 edges except two runs of consecutive disjoint non-edges, one starting at one near-end edge and the other ending with the other near-end edge. (compare with phase 1). We conjecture that the number of non-edges in these runs differs by at most one.

Examples: $1(011)^2 1^2 (011)^2 01$, $1(011)^3 1(011)^2 01$.

3.4 Border cases: There are two border cases. One occurs between phases 1 and 2, and is of the form $(100)^j 101(001)^k$, $j,k \geq 0$, in the case when $n \equiv 0 (\bmod 3)$ and $I=n/3$; the second one is between phases 2 and 3, and is of the form $1(011)^j 010(110)^k 1$, $j,k \geq 0$ (and we conjecture $j-k \leq 1$) when $n \equiv 2 (\bmod 3)$ and $I=(2n-7)/3$. These two configurations resemble phases 1 and 3, respectively, more than phase 2, but from another point of view (the forbidden configurations mentioned in the sequel) they belong to phase 2, and therefore they are discussed in section 6. Note that by exchanging the 0's and 1's in the first border case, and adding 1 in both ends, we get the second border case.

3.5 The structure of the solution is surprisingly complicated, and seems to be a combination of several properties, which clarify it and can help studying the higher levels.

(1) Boundary conditions: Existance of end-edges and absence of near-end edges in the solution.

(2) Locally uniform distribution: The distribution of the edges along the Hamiltonian path is "locally uniform" in some sense. This property is expressed by forbidden configurations for the different phases: 11,101 in phase 1, 111,000,01010,10101 in phase 2, and 00,010 in phase 3.

(3) Runs: The solution contains runs of consecutive disjoint edges.

The solution has also a monotonity property, namely, a solution for m edges is obtainable from that for m-1 edges by adding one more edge. This monotonity holds except for the border case between phases 2 and 3.

3.6 We conclude the description of the solution by presenting the solutions for n=12,13 and 14 and every possible I, partitioned into the phases (* denotes the border cases). The monotonity property is reflected in this example.

	n = 12	n = 13	n = 14	
	100000000	1000000000	10000000000	
	100000001	1000000001	10000000001	
	100100001	1001000001	10010000001	
*	100101001	1001001001	10010001001	
	100101101	1011001001	10010011001	
	101101101	1011011001	10011011001	
	101111101	1011011101	10110101101	*
	111111101	1011111101	10110111101	
	111111111	1111111101	10111111101	
		1111111111	11111111101	
			11111111111	

IV. LOCAL PROPERTIES

4.1 We begin with the following observation. Let G_1 and G_2 be two graphs with I edges in level 3, such that their structure from vertex i to vertex n is the same. The functions $P_1(i)$ and $P_2(i)$ denote the number of paths from vertex 1 to vertex i in G_1 and G_2, respectively, and $\Delta(i) = P_2(i)-P_1(i)$. If for some vertex j, $j \geq i$, we have

$$P_2(j+y) \geq P_1(j+y) \qquad \text{for } y=0,1,2,$$

then $N(G_2) \geq N(G_1)$.

Furthermore, if $c_i=0$ (i.e. $(i,i+3)$ is a non-edge) and

$$P_2(i+2) \geq P_1(i+2) \quad , \quad P_2(i+3) \geq P_1(i+3)$$

then $N(G_2) \geq N(G_1)$. In case $c_{i+1}=0$ this is trivial. Otherwise $P_2(i+3) = P_2(i+2) + P_2(i+1)$ since $c_i=0$. Therefore

$$P_2(i+4) = P_2(i+3) + P_2(i+2) + P_2(i+1) = 2P_2(i+3)$$

$$\geq 2P_1(i+3) = P_1(i+3) + P_1(i+2)+P_1(i+1) = P_1(i+4) .$$

Hence $N(G_2) \geq N(G_1)$ by the first observation.

4.2 Theorem

Every solution contains both end-edges (if I=1 then only one end-edge).

Proof: We prove that the solution contains the edge (1,4). The proof for (n-3,n) follows by symmetry. Suppose that a solution G_1 starts with $0^{i-1}1$, $i>1$. Let G_2 be the graph obtained from G_1 by replacing this configuration by 10^{i-1}.

$P_1(i+3)$ is equal to the number of paths in the opposite direction from i+3 to 1 in G_1. Thus $P_2(i+3) = P_1(i+3)$. It is clear that $P_2(i+2) > P_1(i+2)$ and $P_2(i+1) > P_1(i+1)$. Thus $N(G_2) > N(G_1)$, contradicting the optimality of G_1. Hence G_1 must contain the edge (1,4). □

Corollary: $N_{n,2n-2} = F_n + F_{n-3}$, $N_{n,2n-1} = F_n + 2F_{n-3} + F_{n-6}$.

4.3 Theorem

There exists a solution not containing the near-end edges (unless $I \geq n-4$).

Remark: A stronger version of the theorem is false; for example, 11001 is a solution containing a near-end edge.

Proof: We prove it for the edge (2,5); the proof for (n-4,n-1) follows by symmetry. Suppose that a solution G_1 contains the edge (2,5). Then G_1 begins with $1^{i-1}0$, $i>2$. Let G_2 be the graph we get by replacing this configuration by 101^{i-2} . We compute $\Delta(j) = P_2(j)-P_1(j)$ for $j=i+2,i+3$:

$$\Delta(i+2) = P_2(i+2) - P_1(i+2) = (F^3_{i+2} - F^3_{i-2}) - F^3_{i+2} = -F^3_{i-2} < 0 ,$$

$$\begin{aligned}\Delta(i+3) = P_2(i+3) - P_1(i+3) &= (F^3_{i+3} - F^3_{i-1}) - (F^3_{i+2} + F^3_{i+1}) \\ &= F^3_{i-2} + F^3_{i-3} > 0.\end{aligned}$$

If $c_{i+1}=1$ then $P_1(i+4) = P_2(i+4)$ by symmetry, otherwise

$$\Delta(i+4) = \Delta(i+3) + \Delta(i+2) = F^3_{i-3} .$$

Thus $\Delta(i+4) \geq 0$ in both cases.

$$\Delta(i+5) \geq \Delta(i+4) + \Delta(i+3) + \Delta(i+2)$$

since in case $c_{i+2} = 0$ $\Delta(i+2) < 0$ is omitted. Thus $\Delta(i+5) \geq F^3_{i-3} \geq 0$, and $N(G_2) \geq N(G_1)$, and G_2 is also a solution. □

The next lemma presents inequalities used later.

4.4 Lemma

1. $\frac{3}{2}P(i) \leq P(i+1) \leq 2P(i)$ for $i \geq 2$.
2. $\frac{5}{2}P(i) \leq P(i+2) \leq \frac{11}{3}P(i)$ for $i \geq 3$.
3. $4P(i) \leq P(i+3) \leq \frac{20}{3}P(i)$ for $i \geq 3$.
4. $\frac{5}{3}P(i+2) < \frac{39}{22}P(i+2) \leq P(i+3) \leq 2P(i+2)$ if $c_i=1$, for $i \geq 3$.
5. $P(i+4) = 2(P(i+3)$ iff $c_i=0$ and $c_{i+1}=1$, for $i \geq 1$.

Proof: see Appendix A. □

V. PHASE 1

5.1 We first describe proof techniques, which are used for all three phases, and then demonstrate them in this section for phase 1, which is simpler than both phases 2 and 3.

An essential tool in our proofs is to express the number of paths to any vertex j, in a special configuration of a graph, as a function of P(a) and P(b), where a and b are two vertices at the beginning of this configuration. Such special structures are $(100)^i$, $(110)^i$, $i>0$, and 1^i, $i\geq 3$, for phases 1, 2 and 3, respectively. This enables us to measure the effect of some local transformations on N(G). We use this technique to prove that some configurations are forbidden for the different phases. By eliminating these forbidden configurations we establish some order in the structure of the solution. Thereafter, more local transformations are used to determine the exact structure of the solution. many technically involved calculations were required in the proofs, and only few of them are given in the text; more proof details are found in the Appendices.

5.2 Lemma

The number of paths P(b+j) to the j-th vertex from b, j>1, in a configuration $\underset{abc}{010}0(100)^i$, $i\geq 0$, can be expressed as a function of P(b) and P(c) (and as a function of P(a) and P(b) if the configuration is preceded by a 0), where a,b and c are three consecutive vertices at the beginning of this configuration, as follows:

$$P(b+j) = \begin{cases} 3\cdot 5^{i-1}\cdot z & j = 3i+1 , \quad i \geq 1 , \\ 5^i\cdot z & j = 3i+2 , \quad i \geq 0 , \\ 2\cdot 5^i\cdot z & j = 3i+3 , \quad i \geq 0 , \end{cases}$$

where z = P(b) + P(c) (and, if 0 precedes the configuration, z is also P(a) +2P(b)).

Proof: By a straightforward induction on j, using the corresponding formulae

$P(b+3i+1) = P(b+3i) + P(b+3i-1)$, $P(b+3i+2) = P(b+3i+1) + P(b+3i)$, and $P(b+3i+3) = P(b+3i+2) + P(b+3i+1) + P(b+3i)$. □

We turn now to the structure of phase 1.

5.3 Theorem

In phase 1 of the solution the edges are disjoint (i.e. no two edges intersect).

Proof: We show that if the solution G contains non-disjoint edges, namely G contains either a 11 or a 101 configuration, then it is possible to transform G into a graph G_1 such that $N(G_1) > N(G)$, contradicting the optimality of G.

Since i phase 1 I $\leq \lfloor n/3 \rfloor$, the existance of either 11 or 101 implies the existance of a gap, which corresponds to a 000 configuration, in G. Without loss of generality (wlog) we consider a first occurrence of either 11 or 101, to the left of a 000 configuration.

Case 1: 101.

Wlog we assume the configuration $101(001)^{k-1}000$ in G. The graph G_1 is obtained from G by replacing this configuration by $(100)^{k+1}$.

G :	·	1	0	1	0	0		$(100)^{k-1}$	0	·	·	·	·	·
	u	v	w	a	b				i	i+1				i+5
G_1 :	·	1	0	0	1	0	0	$(100)^{k-1}$		·	·	·	·	·
	u	v	w	a	b					i+1				i+5

We denote $y = P(a) + P(b)$ and $z = P(a) + 2P(b)$ (note that $P_1(a) = P(a)$ and $P_1(b) = P(b)$). P, P_1 and $\Delta = P_1 - P$ for the vertices i - i+5 are summarizes in table 1. The calculations are based on lemmata 4.4 and 5.2, and some of them are given here; the rest, being simpler, are left to the reader.

$\Delta(i+3) = 5 \cdot 5^k \cdot z - 8 \cdot 5^k \cdot y = 5^k\,[-3P(a) + 2P(b)] > 0$, by lemma 4.4(4). If $c_{i+1} = 1$ then lemma 4.4(5) implies that $\Delta(i+4) = 2\,\Delta(i+3) \geq 0$.

	P	P_1	$\Delta = P_1 - P$
i	$2 \cdot 5^k \cdot y$	$5^k \cdot z$	$-5^k P(a) < 0$
$i+1$	$3 \cdot 5^k \cdot y$	$2 \cdot 5^k \cdot z$	$5^k(P(b) - P(a)) > 0$
$i+2$	$5 \cdot 5^k \cdot y$	$3 \cdot 5^k \cdot z$	$5^k(P(b) - 2P(a)) \leq 0$
$i+3$	$8 \cdot 5^k \cdot y$	$5 \cdot 5^k \cdot z$	$5^k(2P(b) - 3P(a)) > 0$
$c_{i+1}=1$, $i+4$	$16 \cdot 5^k \cdot y$	$10 \cdot 5^k \cdot z$	$5^k(4P(b) - 6P(a)) > 0$
$c_{i+1}=0$, $i+4$	$13 \cdot 5^k \cdot y$	$8 \cdot 5^k \cdot z$	$5^k(3P(b) - 5P(a)) > 0$
$c_{i+2}=c_{i+1}=1$, $i+5$	$29 \cdot 5^k \cdot y$	$18 \cdot 5^k \cdot z$	$5^k(11P(a) + 7P(b)) > 0$
$c_{i+2}=1, c_{i+1}=0, i+5$	$26 \cdot 5^k \cdot y$	$16 \cdot 5^k \cdot z$	$5^k(6P(a) - 10P(b)) > 0$

Table 1

If $c_{i+1} = 0$ then $\Delta(i+4) = 8 \cdot 5^k \cdot z - 13 \cdot 5^k \cdot y = 5^k\ [-5P(a)+3P(b)] > 0$, by lemma 4.4(4).

If $c_{i+2} = 0$ then the previous inequlities imply $N(G_1) > N(G)$.

If $c_{i+2} == c_{i+1} = 1$ then $\Delta(i+5) = 18 \cdot 5^k \cdot z - 29 \cdot 5^k \cdot y =$ $5^k\ [-11P(a)+7P(b)] > 5^k\ [6P(b)-10P(a)] > 0$, by lemma 4.4(4). If $c_{i+2} = 1$ and $c_{i+1}=0$ then by lemma 4.4(5) $\Delta(i+5) = 2\Delta(i+4) > 0$, and again $N(G_1) > N(G)$.

Case 2: 11 .

Wlog we assume the configuration $11(001)^k000$ in G. Replacing this configuration by $(100)^{k+1}10$ can be shown to increase N(G), a contradiction. □

From now on we omit long technical proofs in case when the proof technique has already been demonstrated, as we did in case 2 above.

5.4 Theorem

In phase 1 the solution contains exactly two runs of consecutive disjoint edges; one of j edges, $j \geq 1$, starting at one end-edge, and the other of k edges, $k \geq 1$, ending at the other end-edge, where $j + k = I$. i.e. the solution is of the form $(100)^j 0^{n-3-3I}(001)^k$.

Proof: By theorem 5.3 the edges in the solution are disjoint, and by theorem 4.2 both end-edges belong to the solution. We could obtain the solution by applying local transformations and measuring their effect, using previous techniques (similar to the proof technique of theorem 5.3). Instead, we present a different proof, based on a global approach. We could not find such proofs for the other phases, which have a more complicated structure.

We first calculate N(G) for the solution G $(100)^j 0^{n-3-3I}(001)^k$, writing G as

$$\underset{u}{1}\ \underset{v}{0}\ 0\ (100)^{j-2}\ 1\ 0\ \underset{a}{0}\ \underset{b}{0}\ 0^{n-3I+1}\ \underset{c}{1}\ \underset{d}{0}\ 0\ 1\ (001)^{k-2} \ .$$

By lemma 5.2 $P(a) = 5^{j-1}\cdot z$ and $P(b) = 2\cdot 5^{j-1}\cdot z$, where $z = P(u) + P(v) = 1+1 = 2$. Note that the coefficients of $5^{j-1}\cdot z$, 1 and 2 , are the Fibonacci numbers F_2 and F_3 , respectively. Hence

$$P(c) = 2\cdot 5^{j-1}\cdot F_{n-3I+2} \quad \text{and} \quad P(d) = 2\cdot 5^{j-1}\cdot F_{n-3I+3} \ .$$

Then, by lemma 5.2, we get

$$\begin{aligned} P(n) &= 2\cdot 5^{k-1}\cdot (P(c) + P(d)) = 2\cdot 5^{k-1}\cdot 2\cdot 5^{j-1}\cdot (F_{n-3I+2} + F_{n-3I+3}) \\ &= 4\cdot 5^{j+k-2}\cdot F_{n-3I+4} \ , \end{aligned}$$

hence $N(G) = P(n) = 4 \cdot 5^{I-2} \cdot F_{n-3I+4}$.

Suppose G_1 contains I arbitrary disjoint edges: $0^{t_0}10^{t_1}1 \cdots 10^{t_{I-1}}10^{t_I}$,

where $\sum_{i=0}^{I} t_i = n-3-I$. Write G_1 as

$$0^{t_0}\ 1\ 0\ \underset{a}{0}\ \underset{b}{0}\ 0^{t_1-3}\ 1\ 0\ \underset{c}{0}\ \underset{d}{0}\ 0^{t_2-3}\ 1 \cdots 1\ 0^{t_{I-1}}\ 1\ 0\ \underset{e}{0}\ \underset{f}{0}\ 0^{t_I-3}\ .$$

$P(a) = F_{t_0+3}$, and by lemma 4.4(5) $P(b) = 2P(a) = 2F_{t_0+3}$. The coefficients of F_{t_0+3} in P(a) and P(b), namely 1 and 2, are the Fibonacci numbers F_2 and F_3, respectively. Hence $P(c) = F_{t_0+3} \cdot F_{t_1+3}$, and by lemma 4.4(5) we get $P(d) = 2F_{t_0+3} \cdot F_{t_1+3}$. It can be proved by induction on I that

$$P(e) = F_{t_0+3} \cdot F_{t_1+3} \cdot F_{t_2+3} \cdots F_{t_{I-1}+3},$$

$$P(f) = 2P(e) \quad , \quad \text{and}$$

$$N(G) = P(n) = F_{t_0+3} \cdot F_{t_1+3} \cdot F_{t_2+3} \cdots F_{t_{I-1}+3} \cdot F_{t_I+3} .$$

It remains to prove that

$$F_{t_0+3} \cdot F_{t_1+3} \cdots F_{t_{I-1}+3} \cdot F_{t_I+3} \leq 4 \cdot 5^{I-2} \cdot F_{n-3I+4}$$

for any n, $0 < I \leq \left\lfloor \frac{n}{3} \right\rfloor$, and a sequence $\{t_i\}$ satisfying $\Sigma\, t_i = n-3-I$. The proof, by induction on I, can be found in Appendix B. □

<u>Corollary</u>:

$$N_{n,2n-3+I} = 4 \cdot 5^{I-2} \cdot F_{n-3I+4}$$

for $3 \leq I \leq \left\lfloor \frac{n}{3} \right\rfloor$.

VI. PHASE 2

6.1 Lemma

The number of paths $P(a+j)$ to the j-th vertex from a in a configuration $\underset{ab}{0110}(110)^k\cdots$ in G can be expressed as a function of P(a) and P(b), where a and b are two consecutive vertices at the beginning of the configuration, as follows:

$$P(a+j) = a_j\ (\ P(a) + P(b)\ ) - a_{j-3}\ P(a)\ ,$$

where the a_j's satisfy

$$a_0 = 0\ ,\quad a_1 = a_2 = 1\ ,\quad \text{and for}\quad i\geq 1$$

$$a_{3i} = a_{3i-1} + a_{3i-2} + a_{3i-3} = 2a_{3i-1}\ ,$$

$$a_{3i+1} = a_{3i} + a_{3i-1} + a_{3i-2}\ ,$$

$$a_{3i+2} = a_{3i+1} + a_{3i}\ .$$

Proof: by induction. □

Next we will use the fact that in phase 2 $\left\lfloor\frac{n+2}{3}\right\rfloor \leq I \leq \left\lfloor\frac{2n-6}{3}\right\rfloor$ to show forbidden configurations for this phase.

6.2 Theorem

The solution in phase 2 contains no configuration 111 .

Proof: We assume that a solution G contains a configuration 111, and show that it is possible to transform G into G_1 such that $N(G_1)>N(G)$, contradicting the optimality of G. Wlog we consider the rightmost occurrence (in G) of either $111(011)^k00$ or $111(011)^k010$, $k\geq 0$.

Case 1: $111(011)^k00$.

construct G_1 by replacing this configuration by $1(011)^{k+1}0$. Applying lemma 6.1 we calculate the number of paths to two vertices i and i+1 in both G and G_1 .

$$
\begin{array}{lcccccccccccc}
G: & 1 & 1 & 1 & 0 & 1 & 1 & (011)^{k-1} & 0 & 0 & \cdot & \cdot & \cdot \\
 & w & a & b & c & d & e & & & & i & i+1 & i+2
\end{array}
$$

$$
\begin{array}{lccccccccccc}
G_1: & 1 & 0 & 1 & 1 & 0 & 1 & 1 & (011)^{k-1} & 0 & \cdot & \cdot & \cdot \\
 & & a & b & c & & & & & & i & i+1 & i+2
\end{array}
$$

	P	P_1
i	$a_{3k+1}((P(d)+P(e))-a_{3k-2}P(d)$	$a_{3k+3}((p(b)+P(c))-a_{3k}P(b)$
$i+1$	$a_{3k+2}((P(d)+P(e))-a_{3k-1}P(d)$	$a_{3k+4}((P(b)+P(c))-a_{3k+1}P(b)$

$$\underline{\Delta(i)} = a_{3k-1}(P(c)-P(b)-6P(a))-a_{3k-2}P(a) < 0$$

$$\underline{\Delta(i+1)} = a_{3k+1}(P(c)-2P(a))+a_{3k-1}(P(c)+P(b)-3P(a)) > 0$$

Table 2

The details for $\Delta(i)$ and $\Delta(i+1)$ can be found in Appendix C.

$$\Delta(i+2) = \Delta(i+1) + \Delta(i)$$

since $c_{i-1}=0$, hence

$$
\begin{aligned}
\Delta(i+2) &= a_{3k-1}(P(c)-P(b)-6P(a)+P(c)-P(b)-3P(a)) + a_{3k+1}(P(c)-2P(a)) - a_{3k-2}P(a) \\
&= P(c)(a_{3k+1} + 2a_{3k-1}) - P(a)(2a_{3k+1} + 9a_{3k-1} + a_{3k-2}) \\
&= P(c)(a_{3k} + 3a_{3k-1} + a_{3k-2}) - P(a)(2a_{3k} + 11a_{3k-1} + 3a_{3k-2}) \\
&= P(c)(5a_{3k-1} + a_{3k-2}) - P(a)(15a_{3k-1} + 3a_{3k-2}) \\
&= (P(c) - 3P(a))(5a_{3k-1} + a_{3k-2}) \geq 0,
\end{aligned}
$$

since $P(c) \geq 3P(a)$. Then $N(G_1) > N(G)$, since $\Delta(i+1)>0$, $\Delta(i+2)\geq 0$, and $c_{i-1}=0$.

Case 2: $111(011)^k010$.

Replacing this configuration by $11(011)^{k+1}0$ can be shown to increase $N(G)$, a contradiction. □

6.3 Theorem

The solution in phase 2 contains no configuration 000 .

Proof: The proof is similar to that of theorem 6.2 . The computations are even simpler, because the a_i's in the formulae are replaced by coefficients of the form $c \cdot 5^i$ (for a constant c) from phase 1, since configurations $(100)^i$ replace the configurations $(110)^i$ in the analysis. □

6.4 Theorem

The solution in phase 2 contains no configuration 01010 ; this configuration occurs only in the border case between phases 1 and 2, in the case $n \equiv 0 (\mathrm{mod}\ 3)$ and $I=n/3$, where the solution is of the form $(100)^j 101(001)^k$, $j,k \geq 0$.

Proof: Assume that the solution G contains a configuration 01010. Construct a graph G_1 by replacing this configuration with 10010. We compare P and P_1 for several vertices in both G and G_1.

G :	·	·	·	0	1	0	1	0	·	·
	a-3	a-2	a-1	a	b	c			i	i+1
G_1 :	·	·	·	1	0	0	1	0	·	·
	a-3	a-2	a-1	a	b	c			i	i+1

	P	P_1	Δ
i-2	P(b)+P(c)	P(a)+P(b)+P(c)	P(a) > 0
i-1	2(P(b)+P(c))	P(a)+P(b)+2P(c)	P(a)-P(b) < 0
i	3(P(b)+P(c))	2(P(a)+P(b))+3P(c)	2P(a)-P(b) $\geq$ 0

Table 3

By lemma 4.4(5) $\Delta(i+1)=2\Delta(i)$, since $c_{i-3}=0$ and $c_{i-2}=1$ in both graphs. This, together with $c_{i-1}=0$, implies $N(G_1) \geq N(G)$. If $\Delta(i)=2P(a)-P(b) > 0$ then $N(G_1) > N(G)$, contradicting the optimality of G. If $\Delta(i)=0$ and $N(G_1)=N(G)$ then $P(b)=2P(a)$. This, by lemma 4.4(5), implies $c_{a-3}=0$ and $c_{a-2}=1$. Consider the two possibilities for c_{a-1}.

<u>$c_{a-1} = 1$.</u> In this case G_1 contains a 111 starting at a-2, and thus (by theorem 6.2) G_1, and consequently G, are not solutions, a contradiction.

<u>$c_{a-1} = 0$.</u> In this case G and G_1 contain 01001010 and 01010010 starting at a-3, respectively. But now G_1 contains 01010 shifted three positions to the left. Applying the same argument yields the configuration 01001010010 in G_1 (corresponds to 01001001010 in G), and 01010010010 in another solution G_2. By repeatedly applying this argument, and similarly to the right of the original configuration 01010 in G, we can prove the structure $(100)^j101(001)^k$, $j,k \geq 0$, of the border case between phases 1 and 2, as the only exception.

□

<u>6.5</u> <u>Theorem</u>

The solution in phase 2 contains no configuration 10101 ; this configuration occurs only in the border case between phases 2 and 3, in the case $n \equiv 2 \pmod 3$ and $I = \frac{2n-7}{3}$, where the solution is of the form $1(011)^j010(110)^k1$, $j,k \geq 0$.

<u>Proof</u>: Assume a solution G contains 10101. If the solution is not of he desired form then it contains either $11010(110)^k01$, $k \geq 1$, or $[0]1(011)^j010(110)^k10$, $j,k \geq 0$ (or symmetric configurations). It can be shown that replacing these configurations by $(110)^{k+1}1001$ or $[0]100(110)^{j+k+1}$, respectively, increases N(G), a contradiction. □

<u>6.6</u> <u>Conjecture</u>: The solution for the border case between phases 2 and 3 is of the form $10(110)^j1(011)^k01$, $j,k \geq 0$, where $|j-k| \leq 1$. (see also conjecture 7.4)

6.7 Corollary

A solution in phase 2 contains the configuration 111 .

Proof: Follows from theorems 6.3, 6.4 and 6.5. □

In phase 2 the forbidden configurations are not sufficient for a straightforward proof of the structure of the solution, as was the case in phase 1. A detailed "case analysis", given in the following theorem, is required.

6.8 Theorem

The solution in phase 2 contains none of the following configurations $(j,k > 0)$:

a. $0100(110)^{j}[0](100)^{k}1011$

b. $0100(110)^{j}[0](100)^{k}11$

c. $010(110)^{j}[0](100)^{k}1011$

d. $010(110)^{j}[0](100)^{k}11$.

Proof: We prove only (a) (with the optional 0); the rest of the cases are similar. Suppose the solution G contains $0100(110)^{j}0(100)^{k}1011$, $j,k > 0$. Construct G_1 by replacing this configuration with $0(100)^{k+2}(110)^{j}11$. Applying lemmata 5.2 and 6.1 we calculate F and P_1 for several vertices in both G and G_1.

$$
\begin{array}{lccccccccccccccccccc}
G: & 0 & 1 & 0 & 0 & 1 & 1 & 0 & (110)^{j-1} & 0 & 1 & 0 & 0 & (100)^{k-1} & 1 & 0 & 1 & 1 & \cdot & \cdot & \cdot \\
 & & a & b & c & d & e & & & & x & y & & & & & i & i+1 & i+2 & i+3 & i+4
\end{array}
$$

$$
\begin{array}{lcccccccccccccc}
G_1: & 0 & 1 & 0 & 0 & (100)^{k+1} & 1 & 1 & 0 & (110)^{j-1} & 1 & 1 & \cdot & \cdot & \cdot \\
 & & a & b & c & & w & x & & & i & i+1 & i+2 & i+3 & i+4
\end{array}
$$

	P	P_1
w		$2\cdot 5^{k+1}(P(a)+P(b))$
x	$a_{3j+1}(P(d)+P(e))-a_{3j-2}P(d)$	$3\cdot 5^{k+1}(P(a)+P(b))$
y	$a_{3j+2}(P(d)+P(e))-a_{3j-1}P(d)$	
i	$5^k(P(x)+P(y))$	$a_{3j}(P(w)+P(x))-a_{3j-3}P(w)$
i+1	$2\cdot 5^k(P(x)+P(y))$	$a_{3j+1}(P(w)+P(x))-a_{3j-2}P(w)$

$$\underline{\Delta(i)} = 2\cdot 5^k(a_{3j-1} + a_{3j-2})(P(a) + P(b)) > 0$$

$$\underline{\Delta(i+1)} = -5^k(a_{3j-1} + a_{3j-2})(P(a) + P(b)) < 0$$

Table 4

$\Delta(i)$ and $\Delta(i+1)$ are discussed in Appendix D. Let $A = P(a) + P(b)$. Then

$$\Delta(i+2) = \Delta(i+1) + \Delta(i) = 5^k\cdot A\cdot(a_{3j-1} + a_{3j-2}) > 0 , \quad \text{and}$$

$$\Delta(i+3) = 2\Delta(i+2) = 2\cdot 5^k\cdot A\cdot(a_{3j-1} + a_{3j-2}) > 0$$

by lemma 4.4(5). Hence

$$\Delta(i+4) = \Delta(i+3) + \Delta(i+2) + \Delta(i+1) = \Delta(i+3) > 0$$

(since $\Delta(i+2) = -\Delta(i+1)$).

Hence $N(G_1) > N(G)$, a contradiction. $\square$

6.9 Theorem

The solution in phase 2 does not begin with the configuration 1010 and does not end with the configuration 0101 .

Proof: Assume that the solution G begins with 1010. Then G begins with either 1010011 or 1010010, since any other configuration , that starts with 1010, must contain a forbidden configuration. If G starts with 1010011 it can be easily shown that replacing this by 1001011 increases N(G), a contradiction. If G starts with 1010010 it can be easily shown that a graph G_1, obtained from G by replacing this by 1001010, satisfies $N(G_1)=N(G)$. But G_1 is not a solution since it contains the forbidden configuration 01010. Hence G cannot be a solution.

The proof for the second part follows by symmetry. □

6.10 Theorem

There exists a solution for phase 2 of the form

$$(100)^i 10[0](110)^j [0]1(001)^k \ ,$$

$i,j,k \geq 0$. (The options are according to n(mod 3)).

Proof: Consider the rightmost occurrence of 11 in a solution for phase 2. This 11 is followed by $[0]01(001)^k$, $k \geq 0$, since 111,000,01010 are forbidden configurations for phase 2, and 11 cannot appear in this configuration (note that this solution is not unique, since we are using here theorem 4.3 saying that there exists a solution not containing near-end edges). The corresponding analysis applies for the configuration to the left of the leftmost occurrence of 11 in the solution.

Therefore it remains to show that the configuration between the rightmost and the leftmost 11 is $0(110)^{j-2}$. This configuration almost follows from the forbidden configuration shown in theorems 6.2 - 6.5 and 6.8 . But the solution may still contain a 110011. Consider the leftmost occurrence of 110011.

Case 1: The solution starts with $(100)^i10(110)^j0(110)^k$. It can be shown that replacing it by $(100)^{i+1}(110)^{j+k}$ increases N(G), a contradiction.

Case 2: The solution starts with $(100)^i(110)^j0(110)^k$. Consider now the second leftmost 110011. If there is no second 110011 then, by symmetry, the solution is of the form $(100)^i(110)^j(011)^k(001)^t$. But it can be shown that replacing it by $(100)^{i+1}10(110)^{j+k-1}1(001)^{t-1}$ increases N(G), a contradiction. If there exists a second 110011 then the solution starts with $(100)^i(110)^j0(110)^k0(110)^t$. It can be shown that replacing it by $(100)^{i+1}10(110)^{j+k+t-1}$ increases N(G), a contradiction. $\square$

VII. PHASE 3

7.1 Lemma

The number of paths $P(a+j)$ to the j-th vertex from a in a configuration $0\underset{a}{1}\underset{b}{1}1^k$, $k>0$, in a graph G can be expressed as a function of $P(a)$ and $P(b)$, where a and b are the two consecutive vertices at the beginning of the configuration, as follows:

$$P(a+j) = F_j^3(P(a) + P(b)) - F_{j-3}^3 P(a) \quad , \qquad j \geq 2 \, ,$$

where F_t^3 denotes the t-th 3-generalized Fibonacci number, and $F_t^3=0$ for $t \leq 0$.

Proof: By induction on j (see Appendix E).

7.2 Theorem

In the solution in phase 3 the non-edges are disjoint.

Remark: Compare this theorem, and the "case analysis" in its proof, with those of theorem 5.3 in phase 1 (concerning edges rather than non-edges).

Proof: Assume that a solution G contains either 00 or 010.

Case 1: 010 .

Since we are in phase 3 G must contain a 111. Wlog we assume the configuration $010(110)^k111$, $k \geq 0$, in G. Replacing this by $(011)^{k+2}$ increases N(G), (this is shown by applying lemma 6.1), a contradiction.

Case 2: 00 .

As in case 1 we may assume the configuration $00(110)^k111$, $k \geq 0$, in G. Replacing this by $0(110)^{k+1}1$ increases N(G), (as before, by applying lemma 6.1), a contradiction.

□

7.3 Theorem

The solution in phase 3 is composed of two runs of consecutive disjoint non-edges, one starting with one near-end edge and the other ending with the other near-end edge,

$$10(110)^i 1^j (011)^k 01, \quad i,k > 0, \; j \geq 3.$$

Proof: In phase 3 we have a stronger version of theorem 4.3, where we showed that the solution contains both near-end edges. By the previous theorem the non-edges are disjoint, hence the solution contains at least two runs of consecutive disjoint non-edges, one starting with one near-end edge and the other ending with the other near-end edge (these runs may contain only one non-edge). Therefore it remains to show that there are no more runs of consecutive disjoint non-edges.

Suppose G contains more such runs; then G contains at least two runs of consecutive edges (i.e. 1^j for some $j \geq 3$). Hence G contains a configuration $01^j0(110)^k1^t$, $j,t \geq 3$, $k \geq 0$. Construct a graph G_1 by replacing it by $0(110)^{k+1}1^{j+t-2}$. We apply lemmata 6.1 and 7.1 to calculate P and P_1 for several vertices in G and G_1.

G	:	0	1 (a)	1 (b)	1^{j-2}	0	1 (x)	1 (y)	0	$(110)^{k-1}$	1	1	1 (i-2)	. (i-1)	. (i)	. (i+1)	. (i+2)
G_1	:	0	1 (a)	1 (b)	0	$(110)^k$	1 (x)	1 (y)		1^{j-2}			. (i-2)	. (i-1)	. (i)	. (i+1)	. (i+2)

	P	P_1	Δ
x	$F_{j+1}(P(a)+P(b))-F_{j-2}P(a)$	$a_{3k+3}(P(a)+P(b))-a_{3k}P(a)$	
y	$F_{j+2}(P(a)+P(b))-F_{j-1}P(a)$	$a_{3k+4}(P(a)+P(b))-a_{3k+1}P(a)$	
i-2	$a_{3k+2}(P(x)+P(y))-a_{3k-1}P(x)$	$F_j(P_1(x)+P_1(y))-F_{j-3}P(x)$	$F_{j-2}(a_{3k-1}P(b)+(a_{3k+2}+a_{3k+1})P(a))>0$
i-1	$a_{3k+3}(P(x)+P(y))-a_{3k}P(x)$	$F_{j+1}(P_1(x)+P_1(y))-F_{j-2}P(x)$	$-F_{j-3}a_{3k+2}P(a)<0$
i	$a_{3k+4}(P(x)+P(y))-a_{3k+1}P(x)$	$F_{j+2}(P_1(x)+P_1(y))-F_{j-1}P(x)$	$F_{j-3}a_{3k+2}(P(b)-P(a))>0$

Table 5

The details for $\Delta(i) > 0$ are found in Appendix F.

In a way similar to $\Delta(i)$ it can be shown that

$$\Delta(i+1) = F_{j-2}(\ (a_{3k+1}+a_{3k-1})(P(a)+P(b)) + a_{3k-1}P(b)\ ) +$$

$$F_{j-3}(3a_{3k+1} - a_{3k-2})(P(b) - P(a)) > 0 \text{ , and}$$

$$\Delta(i+2) \geq \Delta(i+1) + \Delta(i) + \Delta(i-1) \text{ .}$$

Therefore

$$\Delta(i+2) \geq F_{j-2}(\ (a_{3k+1} + a_{3k-1})(P(a)+P(b)) + a_{3k-1}P(b)\ ) +$$

$$F_{j-3}(3a_{3k+1} - a_{3k-2})(P(b)-P(a))\ ) +$$

$$F_{j-3}a_{3k+2}(P(b)-P(a)) - F_{j-3}a_{3k+2}P(a)$$

$$> F_{j-3}(a_{3k+2} + a_{3k+1} + a_{3k-1})(P(b)-P(a)) + F_{j-3}a_{3k+2}(P(b)-2P(a))$$

$$> F_{j-3}a_{3k+2}(2P(b) - 3P(a)) \geq 0 \text{ , by lemma 4.4(1).}$$

therefore $N(G_1) > N(G)$, a contradiction. □

<u>7.4</u> <u>Conjecture</u>: The numbers of non-edges in the two runs of consecutive disjoint non-edges, in the solution of phase 3, differ by at most one (compare with conjecture 6.6).

We couldn't prove this, since multiplications of a_i's are involved in the computation of $N(G)$; however, experimental results seem to support our conjecture.

APPENDICES

Appendix A

Proof of lemma 4.4:

1. $P(i+1) \leq P(i) + P(i-1) + P(i-2) \leq 2P(i)$.

   $P(i+1) \geq P(i-1) + P(i) \geq \frac{3}{2}P(i)$ by the preceding line.

2. $P(i+2) \geq P(i) + P(i+1) \geq \frac{5}{2}P(i)$ by (1).

   $P(i+2) \leq P(i+1) + P(i) + P(i-1) \leq 2P(i) + P(i) + \frac{2}{3}P(i) = \frac{11}{3}P(i)$ by (1).

3. $P(i+3) \geq P(i+2) + P(i+1) \geq \frac{5}{2}P(i) + \frac{3}{2}P(i) = 4P(i)$.

   $P(i+3) \leq P(i+2) + P(i+1) + P(i) \leq (\frac{11}{3} + 2 + 1)P(i) = \frac{20}{3}P(i)$.

4. $P(i+3) \leq 2P(i+2)$ by (1).

   If $c_i=1$ then

   $P(i+3) \geq P(i+2) + P(i+1) + P(i) \geq \frac{3}{2}P(i+2) + P(i)$ by (1)

   $\geq \frac{3}{2}P(i+2) + \frac{3}{11}P(i+2) = \frac{39}{22}P(i+2)$ by (2) .

5. $P(i+3) = P(i+2) + P(i+1)$ since $c_i=0$, hence

   $P(i+4) = P(i+3) + P(i+2) + P(i+1) = 2P(i+3)$ (since $c_{i+1}=1$),

   and in any other case $P(i+4) < 2P(i+3)$.

Appendix B

End of proof of theorem 5.4:

We prove that

$$F_{t_0+3} \cdot F_{t_1+3} \cdots F_{t_I+3} \leq 4.5^{I-2} \cdot F_{n-3I+4}$$

for any n, $0 < I \leq \lfloor \frac{n}{3} \rfloor$, and the t_i's satisfying $\Sigma t_i = n-3-I$.

Theorem 4.2 implies this inequality for I=1,2 and any n. We assume it holds for I (and any n), and prove it for I+1 (and any n). By the inductive hypothesis we have

$$F_{t_0+3} \cdot F_{t_1+3} \cdots F_{t_I+3} \cdot F_{t_{I+1}+3} \leq 4 \cdot 5^{I-2} \cdot F_{n'-3I+4} \cdot F_{t_{I+1}+3} \quad ,$$

where $n' = n - t_{I+1} - 1$.

Now we use the inequality

$$F_n \cdot F_m \leq F_5 \cdot F_{n+m-5} = 5F_{n+m-5} \qquad \text{for} \quad n,m > 5$$

(which is easily proved by induction), to show that

$$4 \cdot 5^{I-2} \cdot F_{n'-3I+4} \cdot F_{t_{I+1}+3} \leq 4 \cdot 5^{I-1} \cdot F_{n-3I+1} \quad .$$

Hence

$$F_{t_0+3} \cdot F_{t_1+3} \cdots F_{t_I+3} \cdot F_{t_{I+1}+3} \leq 4 \cdot 5^{I-1} \cdot F_{n-3(I+1)+4} \quad .$$

Appendix C

$\Delta(i)$ and $\Delta(i+1)$ in the proof of theorem 6.2:

$$P(d) = P(c) + P(b) + P(a) \ ,$$

$$P(e) = P(d) + P(c) + P(b) = 2P(c) + 2P(b) + P(a) \ ,$$

$$P(e) + P(d) = 3P(c) + 3P(b) + 2P(a).$$

$$\begin{aligned}
\Delta(i) &= P_1(i) - P(i) \\
&= P(c)(a_{3k+3}-3a_{3k+1}+a_{3k-2}) + P(b)(a_{3k+3}-a_{3k}-3a_{3k+1}+a_{3k-2}) + \\
&\quad P(a)(-2a_{3k+1}+a_{3k-2}) \\
&= P(c)(a_{3k+2}-2a_{3k+1}+a_{3k}+a_{3k-2}) + P(b)(a_{3k+2}-2a_{3k+1}+a_{3k-2}) + \\
&\quad P(a)(-2a_{3k}-2a_{3k-1}-a_{3k-2}) \\
&= P(c)(-a_{3k+1}+2a_{3k}+a_{3k-2}) + P(b)(-a_{3k+1}+a_{3k}+a_{3k-2}) + \\
&\quad P(a)(-6a_{3k-1}-a_{3k-2}) \\
&= P(c)(a_{3k}-a_{3k-1}) + P(b)(-a_{3k-1}) + P(a)(-6a_{3k-1}-a_{3k-2}) \\
&= a_{3k-1}(\ P(c)-P(b)-6P(a)\ ) - a_{3k-2}P(a) \\
&= a_{3k-1}(P(w)-5P(a))-a_{3k-2}P(a) \\
&< (-4a_{3k-1}-a_{3k-2})P(a) < 0 \qquad \text{since} \quad P(w) < P(a) \ .
\end{aligned}$$

$$\Delta(i+1)=P(c)(a_{3k+4}-3a_{3k+2}+a_{3k-1}) + P(b)(a_{3k+4}-a_{3k+1}-3a_{3k+2}+a_{3k-1}) + P(a)(-2a_{3k+2}+a_{3k-1})$$

$$=P(c)(a_{3k+3}-2a_{3k+2}+a_{3k+1}+a_{3k-1}) + P(b)(a_{3k+3}-2a_{3k+2}+a_{3k-1}) + P(a)(-2a_{3k+1}-2a_{3k}+a_{3k-1})$$

$$=P(c)(a_{3k+1}+a_{3k-1}) + P(b)a_{3k-1} + P(a)(-2a_{3k+1}-3a_{3k-1})$$

$$=a_{3k+1}(P(c)-2P(a)) + a_{3k-1}(P(c)+P(b)-3P(a)) > 0 \quad , \quad \text{since}$$

$$P(c) = P(b)+P(a)+P(w) \geq 2P(a)+2P(w) \geq 3P(a) \quad \text{by lemma 4.4(1).}$$

Appendix D

$\Delta(i)$ and $\Delta(i+1)$ in the proof of theorem 6.8:

Note that $P(d) = 2(P(a)+P(b))$, $P(e) = 3(P(a)+P(b))$ and $P(d)+P(e) = 5(P(a)+P(b))$, and let $A = P(a)+P(b)$.

$$\Delta(i)= a_{3j}5^{k+2}A - a_{3j-3}2\cdot5^{k+1}A - 5^k(\ (a_{3j+1}+a_{3j+2})5A - (a_{3j-1}+a_{3j-2})2A\ )$$

$$=5^kA(25a_{3j}-10a_{3j-3}-5a_{3j+1}-5a_{3j+2}+2a_{3j-1}+2a_{3j-2})$$

$$=5^kA(-10a_{3j+1}+20a_{3j}+2a_{3j-1}+2a_{3j-2}-10a_{3j-3})$$

$$=5^kA(10a_{3j}-8a_{3j-1}-8a_{3j-2}-10a_{3j-3})$$

$$=5^kA(2a_{3j-1}+2a_{3j-2}) > 0 \ .$$

$$\Delta(i+1)=a_{3j+1}5^{k+2}A - a_{3j-2}2\cdot5^{k+1}A - 2\cdot5^k(\ (a_{3j+1}+a_{3j+2})5A - (a_{3j-1}+a_{3j-2})2A\ )$$

$$=5^kA(25a_{3j+1}-10a_{3j-2}-10a_{3j+1}-10a_{3j+2}+4a_{3j-1}+4a_{3j-2})$$

$$=5^kA(-10a_{3j+2}+15a_{3j+1}+4a_{3j-1}-6a_{3j-2})$$

$$=5^kA(5a_{3j+1}-10a_{3j}+4a_{3j-1}-6a_{3j-2})$$

$$=5^kA(-5a_{3j}+9a_{3j-1}-a_{3j-2})$$

$$=-5^kA(a_{3j-1}+a_{3j-2}) < 0 \quad (\text{since } a_{3j} = 2a_{3j-1}).$$

Appendix E

Proof of lemma 7.1:

$$P(a+2) = P(a) + P(b) \quad \text{and} \quad P(a+3) = 2(P(a) + P(b)).$$

Assume the formula holds for $j<n$. For $j=n$ we get

$$P(a+n) = P(a+n-1) + P(a+n-2) + P(a+n-3)$$

BIBLIOGRAPHIC DATA SHEET	1. Report No. UIUCDCS-R-78-947	2.	3. Recipient's Accession No.
4. Title and Subtitle DEFICIENT GENERALIZED FIBONACCI MAXIMUM PATH GRAPHS			5. Report Date November 1978
			6.
7. Author(s) Y. Perl and S. Zaks			8. Performing Organization Rept. No.
9. Performing Organization Name and Address 222 Digital Computer Lab Dept. of Computer Science University of Illinois Urbana, IL 61801			10. Project/Task/Work Unit No.
			11. Contract/Grant No. NSF MCS 77-22830
12. Sponsoring Organization Name and Address National Science Foundation Washington, D.C.			13. Type of Report & Period Covered
			14.
15. Supplementary Notes			

16. Abstracts

The structure of an acyclic directed graph with n vertices and m edges, maximizing the number of distinct paths between two given vertices, is studied. New techniques for solving this problem are developed, thus enabling us to extend previous results.

17. Key Words and Document Analysis. 17a. Descriptors

Fibonacci graphs, generalized Fibonacci graphs, maximum path graphs.

17b. Identifiers/Open-Ended Terms

17c. COSATI Field/Group

18. Availability Statement	19. Security Class (This Report) UNCLASSIFIED	21. No. of Pages
	20. Security Class (This Page UNCLASSIFIED	22. Price